Descriptive Set Theory
Lecture 5

<u>Properties of Luzin schemes.</u> Let $(A_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ a Luzin scheme of vanishing diameter in a metric space (X,d), and let $f: D \to X$ be the induced map.

(a) f is injective and continuous.

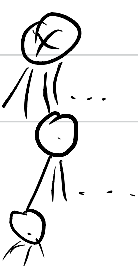
(b) If $A_s = \bigcup_{n\in\mathbb{N}} A_{sn}$ $\forall s\in\mathbb{N}^{<\mathbb{N}}$, then f is surjective.

(c) If A_s is open for each $s\in\mathbb{N}^{<\mathbb{N}}$, then f is open.

(d) If d is complete and $\overline{A_{sn}} \subseteq A_s$ $\forall s\in\mathbb{N}^{<\mathbb{N}}$ and $n\in\mathbb{N}$, then D is closed. In fact, $x\notin D \iff \exists n\ A_{x|n}=\emptyset$.
In particular, if all $A_s\neq\emptyset$, then $D=\mathbb{N}^{\mathbb{N}}$.

Proof. (a) Injectivity is by the fact that if $x\neq y\in D$ then $\exists n$ s.t. $x|n\neq y|n$, but then $A_{x|n}\cap A_{y|n}=\emptyset$.
Continuity is due to vanishing diameter: Fix $x\in D$ and a ball $B(f(x),\varepsilon)$. Then $\exists n$ s.t. $\operatorname{diam}(A_{x|n}) < \varepsilon$ so $A_{x|n}\subseteq B(f(x),\varepsilon)$ since $f(x)\in A_{x|n}$.

(b) For any $y\in X$, one sees by induction on n that $\exists\ s_0 s_1 \dots s_n$ s.t. $y\in A_{s_0 s_1\dots s_n}$. Thus, letting $x:=(s_n)_n \in\mathbb{N}^{\mathbb{N}}$, $f(x)=y$.



(c) Open mean that $f(D$-relatively open$)$ is $f(D)$-relatively open. Because image preserves unions, it's enough to show that $f(\mathcal{U} \cap D)$, for a basic open set $\mathcal{U} = [s]$, $s \in \mathbb{N}^{<\mathbb{N}}$, is open relative to $f(D)$. But $f([s] \cap D) = A_s \cap f(D)$, which is relatively open by the hypothesis.

(d) The facts that d is complete and (A_s) has vanishing diameter implies that $\forall x \in \mathbb{N}^{\mathbb{N}}$, $\bigcap_n \overline{A_{x|n}} \neq \emptyset$ unless $A_{x|n} = \emptyset$ for some n.

But because $\overline{A_{sn}} \subseteq A_s$, $\bigcap_n \overline{A_{x|n}} = \bigcap_n A_{x|n}$, so $\bigcap_n A_{x|n} \neq \emptyset$ unless one of $A_{x|n} = \emptyset$. $\square$

Compact metrizable spaces.

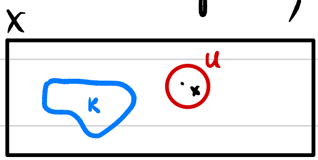
A top. space is compact if every open cover has a finite subcover. By taking complements, we see that this is equivalent to its dual version: every family of closed sets with finite intersection property has a nonempty intersection, where finite intersection property means every finite subcollection has a nonempty intersection.

Some properties related to compactness:

Prop.

(a) Closed subsets of compact spaces are compact.

(b) For Hausdorff spaces, compact $\Rightarrow$ closed.



K compact

$U \cap K = \emptyset$.

(c) Union of finitely many compact subsets is compact.
In particular, finite sets are compact.

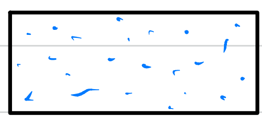
(d) Continuous images of compact sets are compact.
In particular, if $f: X \to Y$ is continuous, X is compact and Y is Hausdorff, then f maps closed to closed.
(Proof. closed in X $\Rightarrow$ compact $\Rightarrow$ f(compact) is compact $\Rightarrow$ closed in Y.)

(e) Continuous injection from a compact space into Hausdorff is in fact an embedding (=: homeomorphism with its image). (Proof. $f: X \hookrightarrow Y$, X compact, Y Hausdorff, then f maps closed subsets to closed subsets. But f is injective, so f-images respect complements, hence f maps open sets to open sets relative to $f(X)$. Thus f is a continuous open bijection between X and $f(X)$, i.e. a homeomorphism.) In particular, a continuous injection of $2^{\mathbb{N}}$ into any Polish space is automatically an embedding.

(f) Disjoint union of finitely many compact spaces is compact.

(g) Tychonoff: products of compact spaces are compact. (Remark. This is equivalent to Axiom of Choice. We'll only use it for ctbl products and that is equivalent to ctbl Choice, which is okay.)

Recall that a metric space (X, d) is called totally bounded if for every $\varepsilon > 0$, there is a finite ε-net, where an ε-net is a subset $S \subseteq X$ s.t. $\forall x \in X \; \exists s \in S$ with $d(s, x) < \varepsilon$ $\iff X \subseteq B(S, \varepsilon) := \bigcup_{s \in S} B(s, \varepsilon)$.



X

S

Prop. Totally bdd metric spaces are separable.

Proof. Let $D_n :=$ a finite $\frac{1}{n}$-net and let $\mathcal{D} := \bigcup_{n \in \mathbb{N}} \mathcal{D}_n$, so $\mathcal{D}$ is ctbl and it is dense b/c $\forall x \in X \; \forall \varepsilon > 0 \; \exists n$ s.t. $\frac{1}{n} < \varepsilon$ and $\exists d \in \mathcal{D}_n$ s.t. $d(x, d) < \varepsilon$. $\square$

Prop. For a metric space (X,d), TFAE:

(1) X is compact.

(2) X is sequentially compact, i.e. every sequence has a convergent subsequence.

(3) Heine–Borel property: X is complete and totally bounded.

The last two propositions imply that compact metrizable spaces are Polish (and all compatible metrics for them are automatically complete).

Examples (of compact metrizable spaces). $2^{\mathbb{N}}$, $\mathbb{T} := S^1 := \mathbb{R}/\mathbb{Z}$, $[0,1]$, $[0,1]^{\mathbb{N}} =:$ Hilbert cube, the space of all probability measures on a compact Polish space, e.g. $[0,1]$, with the weak* topology.

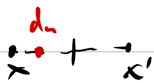
Universality of the Hilbert cube. We now show that $[0,1]^{\mathbb{N}}$ is special among all compact metrizable spaces.

Theorem. Any Polish space is homeomorphic to a G_δ subset of the Hilbert cube and any compact Polish space is homeomorphic to a closed subset of the Hilbert cube.

Proof. The statement about compact spaces follows from the first statement because compact subsets are closed.

Now let X be a Polish space and fix a complete metric $d \leq 1$. Let $D = (d_n)$ be a ctbl dense subset of X and define

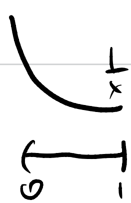
$$f : X \to [0,1]^{\mathbb{N}}$$
$$x \mapsto \big(d(x, d_n)\big)_{n \in \mathbb{N}}.$$

f is injective: . f is continuous because each projection $x \mapsto d(x, d_n)$ is continuous.

$f^{-1} : f(X) \to X$ is continuous because if $f(x_m) \to f(x)$ as $m \to \infty$, then $\forall n$, $d(x_m, d_n) \to d(x, d_n)$ as $m \to \infty$, which implies that $x_m \to x$ as $m \to \infty$ by the density of D. $\square$

Theorem. Each Polish space is homeomorphic to a closed subset of $\mathbb{R}^{\mathbb{N}}$.

Proof. By the previous theorem, enough to show that each G_δ subset $X \subseteq [0,1]^{\mathbb{N}}$ is homeomorphic to a closed subset of $\mathbb{R}^{\mathbb{N}}$. We have already shown that X is homeomorphic to a closed subset of $[0,1]^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \cong \mathbb{R}^{\mathbb{N}}$ and $[0,1]^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ is itself closed in $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$. $\square$

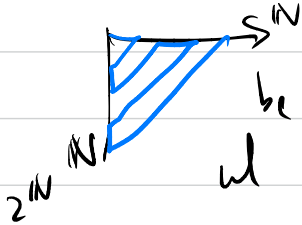


Parametrization of compact metrizable spaces by $2^{\mathbb{N}}$.

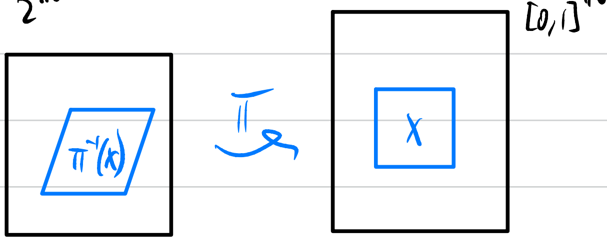
Theorem. Every compact Polish space is a continuous image of $2^{\mathbb{N}}$.

Proof. First, let's do this for $[0,1]$: $f : 2^{\mathbb{N}} \to [0,1]$, $x \mapsto \sum_{n \geq 1} x(n) \cdot 2^{-n}$, which is continuous and surjective.

Thus, $(2^{\mathbb{N}})^{\mathbb{N}} \underset{\text{contin.}}{\twoheadrightarrow} [0,1]^{\mathbb{N}}$ but $(2^{\mathbb{N}})^{\mathbb{N}} \cong 2^{\mathbb{N} \times \mathbb{N}} \cong 2^{\mathbb{N}}$ because $\exists$ bijection $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.



Thus, $\pi : 2^{\mathbb{N}} \underset{\text{cont.}}{\twoheadrightarrow} [0,1]^{\mathbb{N}}$. Now let $X \subseteq [0,1]^{\mathbb{N}}$ be a closed subset, then $\pi^{-1}(X) \subseteq 2^{\mathbb{N}}$ closed and



We let $g : 2^{\mathbb{N}} \to \pi^{-1}(X)$ be a retraction, so $\pi \circ g : 2^{\mathbb{N}} \underset{\text{cont.}}{\twoheadrightarrow} X$. $\square$